Multivariate power series

The PowerSeries package implements multivariate power series over the algebraic closure of the field of the rational numbers.

Creating and accessing power series

PowerSeries is a module which takes an ordered list of the variables. Below, we define multivariate power series in the variables X and Y, for X > Y.

```
PS := PowerSeries([X, Y]) : with(PS);
```

[*Add*, *Coefficient*, *ComputedTerms*, *Eval*, *ExactQuotient*, *GeometricSeries*, *HomogeneousPart*, *HomogeneousPartFunction*, *HomogeneousPartFunctionToPowerSeries*, *Inverse*, *IsUnit*, *Multiply*, *One*, *PolynomialPart*, *PolynomialToPowerSeries*, *Subtract*, *SumOfALLMonomials*, *Zero*] **(1.1.1)**

PowerSeries form a ring (thus have a zero and a one) of which Polynomials in the same variables are a subring. The calculations below show how to instantiate power series . One can see that each power series is represented as a table containing the terms that have been computed so far and a program for computing the next terms; more on this later.

```
s0 := PS:-Zero;
s1 := PS:-One;
op(s1);
s2 := PS:-PolynomialToPowerSeries(1 + X + Y + X^2 + Y^2);
op(s2);
PS:-ComputedTerms(s0);
PS:-ComputedTerms(s1);
PS:-ComputedTerms(s2);
```

power_series

power_series

table([*GEN* = *homog_parts_of_one*, *POLY* = 1, *TYPE* = *power_series*, *DEG* = 3])

power_series

table([*GEN* = *homog_parts_of_poly*, *POLY* = $X^2 + Y^2 + X + Y + 1$, *TYPE* = *power_series*, *DEG* = 2])

0

1

$$X^2 + Y^2 + X + Y + 1 \qquad \textbf{(1.1.2)}$$

The calculations below show a less triavial example (the sum of all monomials) and how the calculated terms of a power series are memorized.

```
s6 := PS:-SumOfALLMonomials;
PS:-ComputedTerms(s6);
g := PS:-HomogeneousPartFunction(s6);
PS:-Coefficient(s6, [1, 1]);
PS:-ComputedTerms(s6);
PS:-HomogeneousPart(s6, 2);
PS:-ComputedTerms(s6);
PS:-HomogeneousPart(s6, 1);
PS:-ComputedTerms(s6);
PS:-HomogeneousPart(s6, 10);
PS:-ComputedTerms(s6);
```

power_series

1

homog_parts_of_sum_of_all_monomials

1

$$X^2 + X Y + Y^2 + X + Y + 1$$

$$X^2 + X Y + Y^2$$

$$X^2 + X Y + Y^2 + X + Y + 1$$

$$X + Y$$

$$X^2 + X Y + Y^2 + X + Y + 1$$

$$X^{10} + X^9 Y + X^8 Y^2 + X^7 Y^3 + X^6 Y^4 + X^5 Y^5 + X^4 Y^6 + X^3 Y^7 + X^2 Y^8 + X Y^9 + Y^{10}$$

$$X^{10} + X^9 Y + X^8 Y^2 + X^7 Y^3 + X^6 Y^4 + X^5 Y^5 + X^4 Y^6 + X^3 Y^7 + X^2 Y^8 + X Y^9 + Y^{10} + X^9 + X^8 Y + X^7 Y^2 + X^6 Y^3 + X^5 Y^4 + X^4 Y^5 + X^3 Y^6 + X^2 Y^7 + X Y^8 + Y^9 + X^8 + X^7 Y + X^6 Y^2 + X^5 Y^3 + X^4 Y^4 + X^3 Y^5 + X^2 Y^6 + X Y^7 + Y^8 + X^7 + X^6 Y + X^5 Y^2 + X^4 Y^3 + X^3 Y^4 + X^2 Y^5 + X Y^6 + Y^7 + X^6 + X^5 Y + X^4 Y^2 + X^3 Y^3 + X^2 Y^4 + X Y^5 + Y^6 + X^5 + X^4 Y + X^3 Y^2 + X^2 Y^3 + X Y^4 + Y^5 + X^4 + X^3 Y + X^2 Y^2 + X Y^3 + Y^4 + X^3 + X^2 Y + X Y^2 + Y^3 + X^2 + X Y + Y^2 + X + Y + 1 \quad \textbf{(1.1.3)}$$

Arithmetic operations on power series

The calculations below show how to addd and subtract power series.

```
s3 := PS:-Add(s1, s2);
PS:-ComputedTerms(s3);
s4 := PS:-Add(s2, s2);
PS:-ComputedTerms(s4);
s5 := PS:-Subtract(s2, s2);
```

power_series

$$X^2 + Y^2 + X + Y + 2$$

power_series

$$2\,X^2 + 2\,Y^2 + 2\,X + 2\,Y + 2$$

power_series **(1.2.1)**

Now, we compute the product of two power series and access the function which computes all the terms of a given total degree.

```
PS:-ComputedTerms(s2);
  s7 := PS:-Multiply(s2, s2);
  g := PS:-HomogeneousPartFunction(s7);
  h0 := g(0);
  h1 := g(1);
  h2 := g(2);
  h3 := g(3);
  h4 := g(4);
  h5 := g(5);
```

$$X^2 + Y^2 + X + Y + 1$$

power_series

homog_parts_of_prod

$$1$$

$$2\,X + 2\,Y$$

$$2\,X^2 + 2\,Y^2 + (X + Y)^2$$

$$2\,(X + Y)\,(X^2 + Y^2)$$

$$(X^2 + Y^2)^2$$

$$0$$ **(1.2.2)**

Every power series with a non-zero constant has an inverse. The calculations below illustrate this fact.

```
s8 := PS:-PolynomialToPowerSeries(1 - X - Y);
  s9 := PS:-ExactQuotient(s1, s8);
  PS:-ComputedTerms(s9);
  PS:-HomogeneousPart(s9, 2);
  PS:-ComputedTerms(s9);
```

$PS{:-}HomogeneousPart(s9, 3);$
$PS{:-}ComputedTerms(s9);$
$PS{:-}HomogeneousPart(s9, 10);$
$PS{:-}ComputedTerms(s9);$
$s10 := PS{:-}Multiply(s8, s9);$
$PS{:-}HomogeneousPart(s10, 10);$

power_series

power_series

$$1$$

$$X^2 + 2\,X\,Y + Y^2$$

$$X^2 + 2\,X\,Y + Y^2 + X + Y + 1$$

$$X^3 + 3\,X^2\,Y + 3\,X\,Y^2 + Y^3$$

$$X^3 + 3\,X^2\,Y + 3\,X\,Y^2 + Y^3 + X^2 + 2\,X\,Y + Y^2 + X + Y + 1$$

$$X^{10} + 10\,X^9\,Y + 45\,X^8\,Y^2 + 120\,X^7\,Y^3 + 210\,X^6\,Y^4 + 252\,X^5\,Y^5 + 210\,X^4\,Y^6 + 120\,X^3\,Y^7 + 45\,X^2\,Y^8 + 10\,X\,Y^9 + Y^{10}$$

$$X^{10} + 10\,X^9\,Y + 45\,X^8\,Y^2 + 120\,X^7\,Y^3 + 210\,X^6\,Y^4 + 252\,X^5\,Y^5 + 210\,X^4\,Y^6 + 120\,X^3\,Y^7 + 45\,X^2\,Y^8 + 10\,X\,Y^9 + Y^{10} + X^9 + 9\,X^8\,Y + 36\,X^7\,Y^2 + 84\,X^6\,Y^3 + 126\,X^5\,Y^4 + 126\,X^4\,Y^5 + 84\,X^3\,Y^6 + 36\,X^2\,Y^7 + 9\,X\,Y^8 + Y^9 + X^8 + 8\,X^7\,Y + 28\,X^6\,Y^2 + 56\,X^5\,Y^3 + 70\,X^4\,Y^4 + 56\,X^3\,Y^5 + 28\,X^2\,Y^6 + 8\,X\,Y^7 + Y^8 + X^7 + 7\,X^6\,Y + 21\,X^5\,Y^2 + 35\,X^4\,Y^3 + 35\,X^3\,Y^4 + 21\,X^2\,Y^5 + 7\,X\,Y^6 + Y^7 + X^6 + 6\,X^5\,Y + 15\,X^4\,Y^2 + 20\,X^3\,Y^3 + 15\,X^2\,Y^4 + 6\,X\,Y^5 + Y^6 + X^5 + 5\,X^4\,Y + 10\,X^3\,Y^2 + 10\,X^2\,Y^3 + 5\,X\,Y^4 + Y^5 + X^4 + 4\,X^3\,Y + 6\,X^2\,Y^2 + 4\,X\,Y^3 + Y^4 + X^3 + 3\,X^2\,Y + 3\,X\,Y^2 + Y^3 + X^2 + 2\,X\,Y + Y^2 + X + Y + 1$$

power_series

$$0 \tag{1.2.3}$$

Another famous example are the geometric series illusrated below.

$f := s1;$
$op(f);$
$h := PS{:-}GeometricSeries;$
$op(h);$
$g := PS{:-}ExactQuotient(f, h);$
$op(g);$
$p := PS{:-}Multiply(h, g);$
$op(p);$
$hp := PS{:-}HomogeneousPart(p, 10);$

```
expand(hp);
```

power_series

$table([GEN = homog\_parts\_of\_one, POLY = 1, TYPE = power\_series, DEG = 10])$

power_series

$table([GEN = homog\_parts\_of\_geom\_series, POLY = 1, TYPE = power\_series, DEG = 0])$

power_series

$table([GEN = homog\_parts\_of\_quotient, POLY = 1, TYPE = power\_series, DEG = 0])$

power_series

$table([GEN = homog\_parts\_of\_prod, POLY = 1, TYPE = power\_series, DEG = 0])$

$$(-X - Y)\,(X^9 + 9X^8Y + 36X^7Y^2 + 84X^6Y^3 + 126X^5Y^4 + 126X^4Y^5 + 84X^3Y^6 + 36X^2Y^7 + 9XY^8 + Y^9) + X^{10} + 10X^9Y + 45X^8Y^2 + 120X^7Y^3 + 210X^6Y^4 + 252X^5Y^5 + 210X^4Y^6 + 120X^3Y^7 + 45X^2Y^8 + 10XY^9 + Y^{10}$$

$$0 \qquad \textbf{(1.2.4)}$$

```
f := s1;
  op(f);
  h := PS:-SumOfALLMonomials;
  op(h);
  g := PS:-ExactQuotient(f, h);
  op(g);
  p := PS:-Multiply(h, g);
  op(p);
  hp := PS:-HomogeneousPart(p, 10);
expand(hp);
```

power_series

$table([GEN = homog\_parts\_of\_one, POLY = 1, TYPE = power\_series, DEG = 10])$

power_series

$table([GEN = homog\_parts\_of\_sum\_of\_all\_monomials, POLY = 1, TYPE = power\_series, DEG = 0])$

power_series

$table([GEN = homog\_parts\_of\_quotient, POLY = 1, TYPE = power\_series, DEG = 0])$

power_series

$table([GEN = homog\_parts\_of\_prod, POLY = 1, TYPE = power\_series, DEG = 0])$

$$(X^8 + X^7 Y + X^6 Y^2 + X^5 Y^3 + X^4 Y^4 + X^3 Y^5 + X^2 Y^6 + X Y^7 + Y^8) X Y + (X^9 + X^8 Y + X^7 Y^2 + X^6 Y^3 + X^5 Y^4 + X^4 Y^5 + X^3 Y^6 + X^2 Y^7 + X Y^8 + Y^9) (-X - Y) + X^{10} + X^9 Y + X^8 Y^2 + X^7 Y^3 + X^6 Y^4 + X^5 Y^5 + X^4 Y^6 + X^3 Y^7 + X^2 Y^8 + X Y^9 + Y^{10}$$

$$0 \qquad \textbf{(1.2.5)}$$

Evaluation of power series

Evaluating a multivariate polynomial at power series is an important operatiion illustrated below.

```
sX := PS:-PolynomialToPowerSeries(X);
  sY := PS:-PolynomialToPowerSeries(Y);
  a := Array([s1, sX, sY]);
  b := [seq(convert('c'[i], symbol), i = 1 ..3)];
  pol := 2 * b[1] - 3 * b[2] + 4 * b[3];
  s13 := Eval(pol, b, a);
 expand(PS:-PolynomialPart(s13, 1) - (2 - 3*X + 4*Y)) ;
```

power_series

power_series

[power_series power_series power_series]

[c[1], c[2], c[3]]

$2\ c[1] - 3\ c[2] + 4\ c[3]$

power_series

$$0 \qquad \textbf{(1.3.1)}$$

```
pol := b[2]^3 * b[3]^2 + RootOf(z^2 + 1, z);
 s14 := Eval(pol, b, a);
 expand(PS:-PolynomialPart(s14, 5) - (RootOf(_Z^2 + 1) + X^3 * Y^2));
```

$c[2]^3\ c[3]^2 + RootOf(\_Z^2 + 1)$

power_series

$$0 \qquad \textbf{(1.3.2)}$$

Univariate polynomials over power series

Univariate polynomials over power series (UPoPS) are an essential tool in algebraic geometry. The package UnivariatePolynomialsOverPowerSeries implement those polynomials in characterisitic zero.

Creating and accessing

UnivariatePolynomialsOverPowerSeries is a module which takes two arguments: the variables of the underlying power series and the variable name of the polynomials. Such a polynomial can be created from an array of its coefficients, thus from an array of power series.

```
PS := PowerSeries([X, Y]) :
UPoPS := UnivariatePolynomialOverPowerSeries([X, Y], Z) :
with(UPoPS);
u1 := UPoPS:-One;
u0 := UPoPS:-Zero;
s1 := PS:-One :
s0 := PS:-Zero :
s2 := PS:-PolynomialToPowerSeries(1 - X);
  s3 := PS:-ExactQuotient(s1, s2);
  hp := PS:-HomogeneousPart(s3, 2);
s4 := PS:-PolynomialToPowerSeries(1 - Y);
  s5 := PS:-Inverse(s4);
  hp := PS:-HomogeneousPart(s5, 3) :
u2 := UPoPS:-FromArrayOfPowerSeries(a);
  UPoPS:-ComputedTerms(u2);
```

[Add, Coefficients, ComputedTerms, ExtendedHenselConstruction, FactorizationViaHenselLemma, FromArrayOfPowerSeries, FromListOfPolynomials, FromPolynomial, LeadingCoefficient, LeadingDegree, MakeMonic, Multiply, One, PolynomialPart, Subtract, Translate, WeierstrassPreparation, Zero]

polynomial_over_power_series

polynomial_over_power_series

power_series

power_series

$$X^2$$

power_series

power_series

polynomial_over_power_series

$$X + (Y^2 + Y + 1)\, Z \qquad (2.1.1)$$

Arit hmetic operations

UPoPS support the usual arithmetic operations (not illustrated below) and translation (illustrated below).

```
u10 := UPoPS:-FromListOfPolynomials([Y, 2, X+1]);
UPoPS:-PolynomialPart(u10, 1);
  u11 := UPoPS:-Translate(u10, -1);
  UPoPS:-PolynomialPart(u11, 1);
 u12 := UPoPS:-Translate(u11, 1);
  UPoPS:-PolynomialPart(u12, 1);
```

polynomial_over_power_series

$$Y+2\,Z+(X+1)\,Z^2$$

polynomial_over_power_series

$$-2\,X\,Z+X+Y-1+(X+1)\,Z^2$$

polynomial_over_power_series

$$Y+2\,Z+(X+1)\,Z^2 \qquad (2.2.1)$$

In particular, one can translate a univariate polynomial over power series at a power series

```
n := 4;
power_series_vars := [seq(convert('X'[i], symbol), i = 0..(n-1))];
  P := PowerSeries(power_series_vars) :
  U := UnivariatePolynomialOverPowerSeries(power_series_vars, Y) :
  u := U:-FromListOfPolynomials([op(power_series_vars), 1]);
  U:-PolynomialPart(u, 1);
  v := U:-Translate(u, 1);
  U:-PolynomialPart(v, 1);
  p := P:-Multiply(P:-PolynomialToPowerSeries(power_series_vars[n]),
            P:-PolynomialToPowerSeries(-1 / n));
  P:-PolynomialPart(p, 1);
  w := U:-Translate(u, p);
  U:-PolynomialPart(w, 3);
 P:-PolynomialPart(p, 1);
  w := U:-Translate(u, p);
  U:-PolynomialPart(w, 3);
```

4

[X[0], X[1], X[2], X[3]]

polynomial_over_power_series

$$X[3]\,Y^3+Y^4+X[2]\,Y^2+X[1]\,Y+X[0]$$

polynomial_over_power_series

$$1 + X[0] + X[1] + X[2] + X[3] + (4 + X[1] + 2\,X[2] + 3\,X[3])\,Y + (6 + X[2] + 3\,X[3])\,Y^2 + (4 + X[3])\,Y^3 + Y^4$$

power_series

$$-\frac{1}{4}\,X[3]$$

polynomial_over_power_series

$$X[0] - \frac{1}{4}\,X[1]\,X[3] + \frac{1}{16}\,X[2]\,X[3]^2 + \left(X[1] - \frac{1}{2}\,X[2]\,X[3] + \frac{1}{8}\,X[3]^3\right)Y + \left(X[2] - \frac{3}{8}\,X[3]^2\right)Y^2 + Y^4$$

$$-\frac{1}{4}\,X[3]$$

polynomial_over_power_series

$$X[0] - \frac{1}{4}\,X[1]\,X[3] + \frac{1}{16}\,X[2]\,X[3]^2 + \left(X[1] - \frac{1}{2}\,X[2]\,X[3] + \frac{1}{8}\,X[3]^3\right)Y + \left(X[2] - \frac{3}{8}\,X[3]^2\right)Y^2 + Y^4 \qquad \textbf{(2.2.2)}$$

Weierstrass preparation factorization

Weierstrass preparation theorem factorizes a UPoPS into a product of an invertible UPoPS (regarded as a power series) and a Weierstrass polynomial (thus a polynomial which vanishes when all variables are specialized to zero). This is an essential tool in the study of analytic functions as well as in factorization algorithms like those based on Hensel lemma.

```
UPoPS:-ComputedTerms(u2);
(p, alpha) := UPoPS:-WeierstrassPreparation(u2, 2);
PS:-ComputedTerms(alpha);
PS:-ComputedTerms(p);
```

$$X + (Y^2 + Y + 1)\,Z$$

polynomial_over_power_series, polynomial_over_power_series

$$Y^2 + Y + 1$$

$$-X\,Y + X + Z \qquad \textbf{(2.3.1)}$$

```
s4 := PS:-PolynomialToPowerSeries(1 - Y);
s5 := PS:-Inverse(s4);
s6 := PS:-PolynomialToPowerSeries(X);
a := Array([s6, s5]);
u2 := UPoPS:-FromArrayOfPowerSeries(a);
UPoPS:-PolynomialPart(u2, 2);
(p, alpha) := UPoPS:-WeierstrassPreparation(u2, 2);
```

```
PS:-ComputedTerms(p);
PS:-ComputedTerms(alpha);
residual := UPoPS:-Subtract(u2, UPoPS:-Multiply(p, alpha));
UPoPS:-PolynomialPart(residual, 2);
```

power_series

power_series

power_series

[power_series   power_series]

polynomial_over_power_series

$$X + (Y^2 + Y + 1) Z$$

polynomial_over_power_series, polynomial_over_power_series

$$-X Y + X + Z$$

$$Y^2 + Y + 1$$

polynomial_over_power_series

$$0 \quad \textbf{(2.3.2)}$$

The example below shows that the invertible UPoPS may be a UPoPS of positive degree.

```
u9 := UPoPS:-FromListOfPolynomials([Y, 1, X]);
UPoPS:-PolynomialPart(u9, 2);
(p, alpha) := UPoPS:-WeierstrassPreparation(u9, 2);
UPoPS:-PolynomialPart(p, 2);
UPoPS:-PolynomialPart(alpha, 2);
(p, alpha) := UPoPS:-WeierstrassPreparation(u9, 4);
UPoPS:-PolynomialPart(p, 4);
UPoPS:-PolynomialPart(alpha, 4);
residual := UPoPS:-Subtract(u9, UPoPS:-Multiply(p, alpha));
UPoPS:-PolynomialPart(residual, 4);
```

polynomial_over_power_series

$$X Z^2 + Y + Z$$

polynomial_over_power_series, polynomial_over_power_series

$$Y + Z$$

$$-X Y + X Z + 1$$

polynomial_over_power_series, polynomial_over_power_series

$$X Y^2 + Y + Z$$

$$-X^2 Y^2 - X Y + X Z + 1$$

polynomial_over_power_series

$$0 \qquad (2.3.3)$$

The example below shows that, in the Weierstrass factorization, the invertible UPoPS and the Weieretrsass factor may be a UPoPS of positive degree, even if the input UPoPS was already invertible (as a power series).

```
u10 := UPoPS:-FromListOfPolynomials([Y, 1, X+1]);
  UPoPS:-PolynomialPart(u10, 2);
  (p, alpha) := UPoPS:-WeierstrassPreparation(u10, 2);
  UPoPS:-PolynomialPart(p, 2);
  UPoPS:-PolynomialPart(alpha, 2);
  residual := UPoPS:-Subtract(u10, UPoPS:-Multiply(p, alpha));
  UPoPS:-PolynomialPart(residual, 2);
 (p, alpha) := UPoPS:-WeierstrassPreparation(u10, 4);
  UPoPS:-PolynomialPart(p, 4);
  UPoPS:-PolynomialPart(alpha, 4);
  residual := UPoPS:-Subtract(u10, UPoPS:-Multiply(p, alpha));
  UPoPS:-PolynomialPart(residual, 4);
```

$$polynomial\_over\_power\_series$$

$$Y + Z + (X+1) Z^2$$

$$polynomial\_over\_power\_series, polynomial\_over\_power\_series$$

$$Y^2 + Y + Z$$

$$-X Y - Y^2 - Y + 1 + (X+1) Z$$

$$polynomial\_over\_power\_series$$

$$0$$

$$polynomial\_over\_power\_series, polynomial\_over\_power\_series$$

$$4 X Y^3 + 5 Y^4 + X Y^2 + 2 Y^3 + Y^2 + Y + Z$$

$$-2 X Y^2 - 2 Y^3 - X Y - Y^2 - Y + 1 - \left(X Y^2 + 2 Y^3\right) X - 4 X Y^3 - 5 Y^4 + (X + 1) Z$$

$$polynomial\_over\_power\_series$$

$$0 \qquad (2.3.4)$$

Hensel lemma factorization

Univariate polynomials over power series (UPoPS) form a unique factorization domain. Hensel lemma leads to a factorization algorithm for monic UPoPS.

```
f := UPoPS:-FromListOfPolynomials([-1, -X, 1]);
 UPoPS:-PolynomialPart(f, 2);
  facts := FactorizationViaHenselLemma(f, 2, true, true);
  map(PolynomialPart, facts, 2);
```

polynomial_over_power_series

$$-X Z+Z^2-1$$

[*polynomial_over_power_series*, *polynomial_over_power_series*]

$$\left[-1-\frac{1}{2} X-\frac{1}{8} X^2+Z, 1-\frac{1}{2} X+\frac{1}{8} X^2+Z\right] \tag{2.4.1}$$

```
f := UPoPS:-FromListOfPolynomials([2, -1-X, -2+X, 1]);
UPoPS:-PolynomialPart( f, 2);
  facts := FactorizationViaHenselLemma(f, 10, true, true);
  map(PolynomialPart, facts, 10);
```

polynomial_over_power_series

$$2+(-1-X) Z+(-2+X) Z^2+Z^3$$

[*polynomial_over_power_series*, *polynomial_over_power_series*, *polynomial_over_power_series*]

$$\left[-2+\frac{2}{3} X-\frac{2}{27} X^2-\frac{2}{243} X^3+\frac{2}{2187} X^4+\frac{10}{19683} X^5+\frac{2}{59049} X^6 -\frac{14}{531441} X^7-\frac{34}{4782969} X^8+\frac{82}{129140163} X^9+\frac{782}{1162261467} X^{10} +Z, Z-1, 1+\frac{1}{3} X+\frac{2}{27} X^2+\frac{2}{243} X^3-\frac{2}{2187} X^4-\frac{10}{19683} X^5 -\frac{2}{59049} X^6+\frac{14}{531441} X^7+\frac{34}{4782969} X^8-\frac{82}{129140163} X^9 -\frac{782}{1162261467} X^{10}+Z\right] \tag{2.4.2}$$

```
f := UPoPS:-FromListOfPolynomials([1, -X, -Y, 1]);
UPoPS:-PolynomialPart( f, 2);
  facts := FactorizationViaHenselLemma(f, 2, true, true);
  map(PolynomialPart, facts, 2);
```

polynomial_over_power_series

$$-Y Z^2+Z^3-X Z+1$$

[*polynomial_over_power_series*, *polynomial_over_power_series*, *polynomial_over_power_series*]

$$\left[RootOf(\_Z^2-\_Z+1)-1-\frac{1}{3} RootOf(\_Z^2-\_Z+1) X-\frac{1}{3} Y -\frac{1}{9} X Y RootOf(\_Z^2-\_Z+1)+\frac{1}{9} X Y-\frac{1}{9} RootOf(\_Z^2-\_Z+1) Y^2 +Z, -RootOf(\_Z^2-\_Z+1)+\frac{1}{3} RootOf(\_Z^2-\_Z+1) X-\frac{1}{3} X-\frac{1}{3} Y +\frac{1}{9} X Y RootOf(\_Z^2-\_Z+1)+\frac{1}{9} RootOf(\_Z^2-\_Z+1) Y^2-\frac{1}{9} Y^2\right. \tag{2.4.3}$$

$+ Z, 1 + \frac{1}{3} X - \frac{1}{3} Y - \frac{1}{9} X Y + \frac{1}{9} Y^2 + Z\Big]$

Extended Hensel construction

For a univariate polynomiial over univariate power series, the extended Hensel construction (EHC) produces the same result as Newton-Puiseux algorithm, thus factoring such univariate polynomial into degree-one polynomials with Puiseux series coefficients.

```
P := PowerSeries([y]) :
U := UnivariatePolynomialOverPowerSeries([y], x) :
poly := y*x^3  + (-2*y + 1)*x + y;
OutputFlag :: name :='parametric':
parametricVar :: name := T:
iter := 3 :
verificationFlag :: boolean := true:
U:-ExtendedHenselConstruction(poly, 0, iter, OutputFlag, parametricVar,
    verificationFlag);
```

$$y x^3 + (-2\, y + 1)\, x + y$$

$$\left[[y = T^2, x = -T^3], \left[y = T^2, x = \frac{RootOf(\_Z^2 + 1)\; T - T^3\; RootOf(\_Z^2 + 1) + \frac{1}{2}\; T^4}{T} \right], \left[y = T^2, x = \frac{-RootOf(\_Z^2 + 1)\; T + T^3\; RootOf(\_Z^2 + 1) + \frac{1}{2}\; T^4}{T} \right] \right] \quad \textbf{(2.5.1)}$$

For a univariate polynomial over multivariate power series, the extended Hensel construction (EHC) factors also this univariate polynomial into degree-one polynomials with Puiseux series coefficients (over the algebraic closure of a rational function field.) This can be understood as a weak version of Jung-Abhyankar Theorem.

```
P := PowerSeries([y, z]) :
U := UnivariatePolynomialOverPowerSeries([y, z], x) :
poly := y*x^3  + (-2*y + z + 1)*x + y;
U:-ExtendedHenselConstruction(poly, [0, 0], 3);
```

$$y x^3 + (-2\, y + z + 1)\, x + y$$

$$\Big[[x = -y], \Big[x \quad \textbf{(2.5.2)}$$

$$= \frac{1}{y}\left(RootOf(\_Z^2+y) - y\,RootOf(\_Z^2+y) + \frac{1}{2}\,RootOf(\_Z^2+y)\,z + \frac{1}{2}\,y^2\right)\Bigg], \Bigg[x$$

$$= \frac{1}{y}\left(-RootOf(\_Z^2+y) + y\,RootOf(\_Z^2+y) - \frac{1}{2}\,RootOf(\_Z^2+y)\,z + \frac{1}{2}\,y^2\right)\Bigg]\Bigg]$$