Solving integer points with Polyhedra library

1. Introduction

The Polyhedra library offers commands to compute the integer solutions to a linear system. The input system contains linear equations, as well as linear inequalities. For a given linear system, the coefficients must be in rational number field. The options for the input are: darkshadow, greyshadow, HNF, partition. We will see how they work later.

Define the polyhedra model with the variables x, y, z, w :

```
> restart:
  PLHD := Polyhedra([x,y,z,w]);with(PLHD);
```

$$PLHD := \textbf{module}(\,)\ \ldots\ \textbf{end module}$$

$$[Canonicalize, DarkShadow, GreyShadow, IntegerSolve, RealShadow] \qquad \textbf{(1.1)}$$

2. Functions

This section introduces the user to the module of Polyhedra library.

2.1 Canonicalize

The input system consists of equations, inequalities and the boolean value HNF with default value true. It aims to solve the integer solutions of the input equations, as well the implicit equations in the inequality system. If HNF is set to be true, this function will use the HNF method, otherwise, it will use the Pugh's method.
Moreover, after replacing the solved equations to the inequality system, this function will simplify the new inequality system to a triangularized form, which has no implicit equations and redundant inequalities. Each inequality in this triangularized form has content 1.
Finally, this function will output the solved equations and the simplified inequalities.

```
> equations := [];
  inequalities := [ 2 * x+3 * y-4 * z+3 * w <=  1,
        -2 * x-3 * y+4 * z-3 * w <=  -1,
        -13 * x-18 * y+24 * z-20 * w  <=  -1,
        -26 * x-40 * y+54 * z-39 * w  <=  0,
        -24 * x-38 * y+49 * z-31 * w  <=  5,
        54 * x+81 * y-109 * z+81 * w  <=  2];
```

$$equations := [\,]$$

$$inequalities := [2x+3y-4z+3w \le 1,\ -2x-3y+4z-3w \le -1,\ -13x-18y+24z-20w \le -1,\ -26x-40y+54z-39w \le 0,\ -24x-38y+49z-31w \le 5,\ 54x+81y-109z+81w \le 2] \qquad \textbf{(2.1.1)}$$

```
> canon := Canonicalize(equations, inequalities, true);
```

$$canon := [[w = 1-2\,t\_1-t\_2,\ x = -1+3\,t\_1+2\,t\_3,\ y = t\_2,\ z = t\_3],\ [\qquad \textbf{(2.1.2)}$$

$-t\_3 \le -25,\ t\_3 \le 34,\ 2\,t\_3 - t\_2 \le 13,\ 13\,t\_2 - 19\,t\_3 \le 72,\ -7\,t\_2 + t\_3 - 10\,t\_1 \le 12,\ t\_1 - 2\,t\_3 + 2\,t\_2 \le 6]]$

```
> canon := Canonicalize(equations, inequalities, false);
```

$$canon := [[y = -2\,t\_0 + 2\,z - w + 1,\ x = 3\,t\_0 - z - 1],\ [-z \le -25,\ z \le 34,\ 2\,z - w \le 50,\ 13\,w - 5\,z \le 73,\ -3\,t\_0 + z - 2\,w \le 4,\ 2\,t\_0 + w \le 14,\ 4\,t\_0 - 3\,z + 7\,w \le 19]] \quad \textbf{(2.1.3)}$$

2.2 DarkShadow

The input system consist of the inequality system, the variable list and the boolean values HNF, partition and recursive, all with default values true.
This function aims to give the representation of the dark shadow of the inequality system w.r.t. the first variable in the variable list.
If recursive is set to be false, the output will consist of one part, which consist of equations, inequalities and variable list.
If recursive is set to be true, this function will call IntegerSolve to the above equations, inequalities, with dark_shadow and grey_shadow both equal to true, HNF and partition the same as inputs.

```
> equations := []; inequalities := [3*x-2*y+z<= 7, -2*x+2*y-z
  <= 12, -4*x+y+3*z <= 15, -y <= -25];
```

$$equations := [\,]$$

$$inequalities := [3\,x - 2\,y + z \le 7,\ -2\,x + 2\,y - z \le 12,\ -4\,x + y + 3\,z \le 15,\ -y \le -25] \quad \textbf{(2.2.1)}$$

```
> DS := DarkShadow(inequalities, [x,y,z], true, true, true);
```

$$DS := [[[\,], [-z \le -2,\ z \le 17,\ -y \le -25,\ -5\,y + 13\,z \le 67,\ 2\,y - z \le 48],\ [y, z]]] \quad \textbf{(2.2.2)}$$

```
> DS := DarkShadow(inequalities, [x,y,z], true, true, false);
```

$$DS := [[[\,], [-z \le -2,\ z \le 17,\ -y \le -25,\ -5\,y + 13\,z \le 67,\ 2\,y - z \le 48]],\ [y, z]] \quad \textbf{(2.2.3)}$$

2.3 GreyShadow

The input system consist of the inequality system, the variable list and the boolean values HNF, partition and cleanup, all with default values true.
This function aims to partition the grey shadow w.r.t. the first variable in the variable list. Each part of the output system consist of the solved equations and the inequalities.
If HNF is set to be true, the solved equations is obtained by HNF method, otherwise, by Pugh's method.
If partition is set to be true, each integer point in the grey shadow must be in exactly one part of the output system. Othewise, there may exist integer point in the grey shadow which satisfies several parts of the output system at the same time.

The boolean value recursive is similar to what we have introduced in Section 2.2.

```
> GS := GreyShadow(inequalities, [x,y,z], true, true, true);
```

$$GS := [[[y = 4x - 3z + 15], [-z \le -15, z \le 18, -4x + 3z \le -10, -5x + 7z \le 37, 5x - 7z \le -36], [x, y, z]], [[y = 27], [-z \le -15, z \le 18, -4x + 3z \le -10, -5x + 7z \le 37, 5x - 7z \le -36, 6x - 7z \le -18], [y, x, z]], [[y = 33], [-z \le -15, z \le 18, -4x + 3z \le -10, -5x + 7z \le 37, 5x - 7z \le -36, 6x - 7z \le -18], [y, x, z]], [[y = 25, x = 14, z = 15], [\,], [x, y, z]], [[z = -50 + 2y, x = 19], [-y \le -25, y \le 34], [x, y, z]]] \quad (2.3.1)$$

```
> GS := GreyShadow(inequalities, [x,y,z], true, true, false);
```

$$GS := [[[y = 4x - 3z + 15], [-z \le -15, z \le 18, -4x + 3z \le -10, -5x + 7z \le 37, 5x - 7z \le -36, 6x - 7z \le -18]], [[y = 25, x = 14, z = 15, t\_1 = 17], [\,]], [[z = -50 + 2y, x = 19], [-y \le -25, y \le 34]]] \quad (2.3.2)$$

```
> GS := GreyShadow(inequalities, [x,y,z], true, false, true);
```

$$GS := [[[y = 4x - 3z + 15], [-z \le -15, z \le 16, -4x + 3z \le -10, -5x + 7z \le 37, 6x - 7z \le -18], [x, y, z]], [[y = 33], [-z \le -14, z \le 18, -4x + 3z \le -10, -5x + 7z \le 37, 6x - 7z \le -18], [y, x, z]], [[y = 25], [-z \le -14, z \le 18, -4x + 3z \le -10, -5x + 7z \le 37, 6x - 7z \le -18], [y, x, z]], [[y = 4x - 31], [-z \le -13, z \le 18, -4x + 3z \le -11, -5x + 7z \le 35, 6x - 7z \le -16], [x, y, z]], [[y = 31], [-z \le -13, z \le 18, -4x + 3z \le -11, -5x + 7z \le 35, 6x - 7z \le -16], [y, x, z]], [[y = 30], [-z \le -13, z \le 18, -4x + 3z \le -11, -5x + 7z \le 35, 6x - 7z \le -16], [y, x, z]], [[z = -2x + 2y - 12], [-y \le -25, y \le 34, -10x + 7y \le 51, x \le 19], [x, y, z]]] \quad (2.3.3)$$

```
> GS := GreyShadow(inequalities, [x,y,z], true, false, false);
```

$$GS := [[[y = 4x - 3z + 15], [-z \le -14, z \le 18, -4x + 3z \le -10, -5x + 7z \le 37, 6x - 7z \le -18]], [[y = 4x - 3z + 14], [-z \le -13, z \le 18, -4x + 3z \le -11, -5x + 7z \le 35, 6x - 7z \le -16]], [[z = -2x + 2y - 12], [-y \le -25, y \le 34, -10x + 7y \le 51, x \le 19]]] \quad (2.3.4)$$

2.4 RealShadow

This function mainly uses the command RegularChains:-SemiAlgebraicSetTools:-LinearSolve to get the simplified representation of the real shadow.

```
> RS := RealShadow(equations, inequalities);
```

$$RS := \left[0 \le z, z \le \frac{132}{7}, 25 \le y, -\frac{73}{5} + \frac{13}{5}z \le y, y \le 25 + \frac{1}{2}z\right] \quad (2.4.1)$$

2.5 Plot

This section will plot the input system, the dark shadow and real shdow to see

their difference.

```
> with(PolyhedralSets):
  sys := [op(equations), op(inequalities)];
  SP := PolyhedralSet(sys);
  DP := PolyhedralSet([op(DS[1,1]), op(DS[1,2]), x=0], [x,y,z]
  ) ;
  RP := PolyhedralSet([op(RS), x=0], [x,y,z]);
  point_in_DP := PolyhedralSet([x=0, y=29, z=9]);
  Region_in_sys := PolyhedralSet([op(sys), y=29, z=9]);
  Plot([SP, DP, RP, point_in_DP, Region_in_sys], color=[blue,
  blue, grey, red, red], transparency = [0.8, 0.5, 0.8, 0.5,
  0.5]);
```

$$sys := [3x - 2y + z \le 7,\ -2x + 2y - z \le 12,\ -4x + y + 3z \le 15,\ -y \le -25]$$

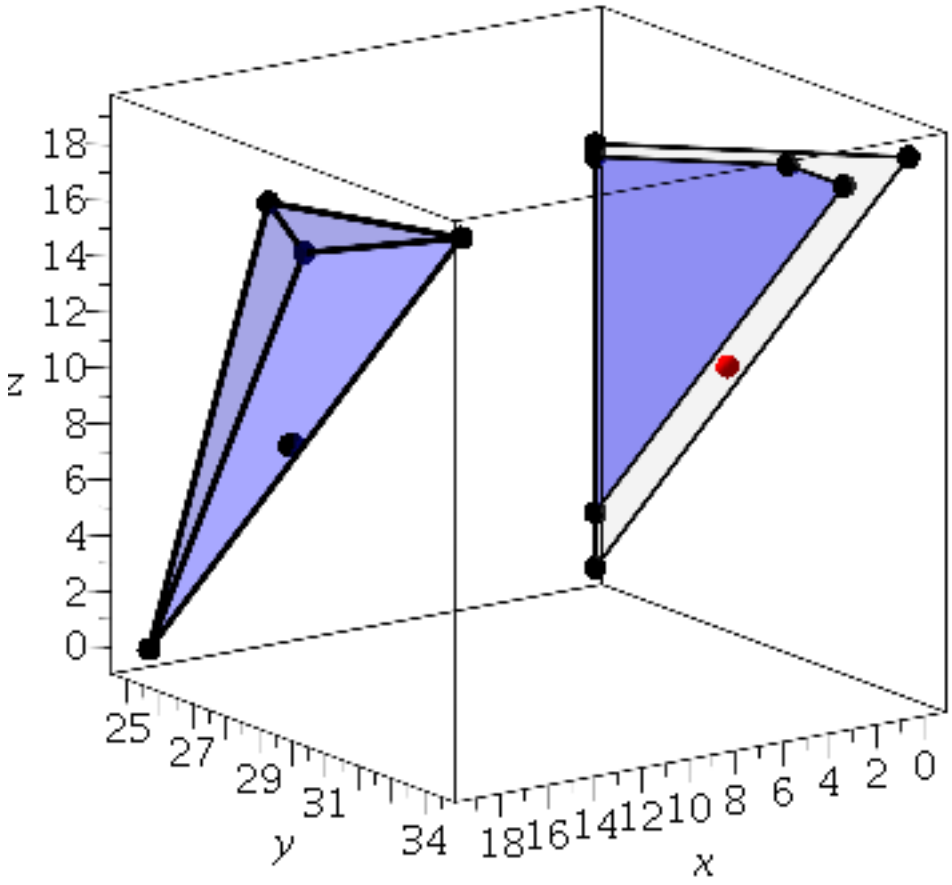
$$SP := \begin{cases} Coordinates & : & [x, y, z] \\ Relations & : & \left[-y \le -25,\ -x + \frac{y}{4} + \frac{3z}{4} \le \frac{15}{4},\ -x + y - \frac{z}{2} \le 6,\ x - \frac{2y}{3} + \frac{z}{3} \le \right. \end{cases}$$

$$DP := \begin{cases} Coordinates & : & [x, y, z] \\ Relations & : & \left[z \le 17,\ -y \le -25,\ -y + \frac{13z}{5} \le \frac{67}{5},\ y - \frac{z}{2} \le 24,\ x = 0\right] \end{cases}$$

$$RP := \begin{cases} Coordinates & : & [x, y, z] \\ Relations & : & \left[-y \le -25,\ -y + \frac{13z}{5} \le \frac{73}{5},\ y - \frac{z}{2} \le 25,\ x = 0\right] \end{cases}$$

$$point\_in\_DP := \begin{cases} Coordinates & : & [x, y, z] \\ Relations & : & [z = 9,\ y = 29,\ x = 0] \end{cases}$$

$$Region\_in\_sys := \begin{cases} Coordinates & : & [x, y, z] \\ Relations & : & \left[z = 9,\ y = 29,\ -x \le -\frac{37}{2},\ x \le \frac{56}{3}\right] \end{cases}$$



We can see the point (y, z) = (29, 9) is in the real shadow, but not in the dark shadow.
Substitute (y, z) = (29, 9) to the input system, we get 37/2 <= x <= 56/3, which has no integer point included.

2.6 IntegerSolve

This section will introduce the main command IntegerSolve.
The input system consists of equations, inequalities, the boolean values dark_shadow, grey_shadow, HNF and partition, all with default values true.
Each part of the output consists of solved equations, inequalities and the variable list of this part.
This function recursively calls the command DarkShadow and GreyShadow.
If we turn off dark_shadow (set it to be false), this function will only implement the GreyShadow. Same with the grey_shadow.
The option HNF is to choose the methods (HNF or Pugh's method) to solve the equations.
The options partition is to determine whether we want a disjoint decomposition of the integer points or not.

```
> IntegerSolve(equations, inequalities, true, true, true,
```

```
  true);
```

$$[[[\,],[-z\le -2,\ z\le 17,\ -y\le -25,\ -5y+13z\le 67,\ 2y-z\le 48,\ -4x+y+3z\le 15,\ -2x+2y-z\le 12,\ 3x-2y+z\le 7],[x,y,z]],[[y=4x-3z+15],[-z\le -15,\ z\le 18,\ -4x+3z\le -10,\ -5x+7z\le 37,\ 5x-7z\le -36],[x,y,z]],[[y=27,\ z=16,\ x=15],[\,],[x,y,z]],[[y=33,\ z=18,\ x=18],[\,],[x,y,z]],[[y=25,\ x=14,\ z=15],[\,],[x,y,z]],[[z=-50+2y,\ x=19],[-y\le -25,\ y\le 34],[x,y,z]]]$$ **(2.6.1)**

```
> unassign('x', 'y', 'z', 'w');IntegerSolve(equations,
  inequalities, true, true, false, true);
```

$$[[[\,],[-z\le -2,\ z\le 17,\ -y\le -25,\ -5y+13z\le 67,\ 2y-z\le 48,\ -4x+y+3z\le 15,\ -2x+2y-z\le 12,\ 3x-2y+z\le 7],[x,y,z]],[[y=4x-3z+15],[-z\le -15,\ z\le 18,\ -4x+3z\le -10,\ -5x+7z\le 37,\ 5x-7z\le -36],[x,y,z]],[[y=27,\ z=16,\ x=15],[\,],[x,y,z]],[[y=33,\ z=18,\ x=18],[\,],[x,y,z]],[[y=25,\ z=15,\ x=14],[\,],[x,y,z]],[[z=-50+2y,\ x=19],[-y\le -25,\ y\le 34],[x,y,z]]]$$ **(2.6.2)**

```
> unassign('x', 'y', 'z', 'w');IntegerSolve(equations,
  inequalities, true, true, true, false);
```

$$[[[\,],[-z\le -2,\ z\le 17,\ -y\le -25,\ -5y+13z\le 67,\ 2y-z\le 48,\ -4x+y+3z\le 15,\ -2x+2y-z\le 12,\ 3x-2y+z\le 7],[x,y,z]],[[y=31,\ z=17,\ x=17],[\,],[x,y,z]],[[y=4x-3z+15],[-z\le -15,\ z\le 16,\ -4x+3z\le -10,\ -5x+7z\le 37,\ 6x-7z\le -18],[x,y,z]],[[y=27],[-z\le -15,\ z\le 16,\ -4x+3z\le -12,\ 6x-7z\le -18,\ 3x+z\le 61],[y,x,z]],[[y=33,\ z=18,\ x=18],[\,],[x,y,z]],[[y=26],[-z\le -14,\ z\le 15,\ -4x+3z\le -11,\ 6x-7z\le -18,\ 3x+z\le 59],[y,x,z]],[[y=25],[-z\le -14,\ z\le 15,\ -4x+3z\le -10,\ 6x-7z\le -18,\ 3x+z\le 57],[y,x,z]],[[y=26],[-z\le -14,\ z\le 15,\ -4x+3z\le -11,\ 6x-7z\le -18,\ 3x+z\le 59],[y,x,z]],[[y=27],[-z\le -15,\ z\le 16,\ -4x+3z\le -12,\ 6x-7z\le -18,\ 3x+z\le 61],[y,x,z]],[[y=25,\ z=15,\ x=14],[\,],[x,y,z]],[[y=25],[-z\le -13,\ z\le 15,\ -4x+3z\le -11,\ 6x-7z\le -16,\ 3x+z\le 57],[y,x,z]],[[y=31,\ z=17,\ x=17],[\,],[x,y,z]],[[y=30,\ z=16,\ x=16],[\,],[x,y,z]],[[y=25],[-z\le -13,\ z\le 15,\ -4x+3z\le -11,\ 6x-7z\le -16,\ 3x+z\le 57],[y,x,z]],[[z=-2x+2y-12],[-y\le -25,\ y\le 34,\ -10x+7y\le 51,\ y-x\le 15,\ x\le 19],[x,y,z]]]$$ **(2.6.3)**

```
> unassign('x', 'y', 'z', 'w');IntegerSolve(equations,
  inequalities, true, true, false, false);
```

$$[[[\,],[-z\le -2,\ z\le 17,\ -y\le -25,\ -5y+13z\le 67,\ 2y-z\le 48,\ -4x+y+3z\le 15,\ -2x+2y-z\le 12,\ 3x-2y+z\le 7],[x,y,z]],[[z=17,\ y=31,\ x=17],[\,],[x,y,z]],[[y=4x-3z+15],[-z\le -15,\ z\le 16,$$ **(2.6.4)**

$$-4\,x+3\,z\le -10,\ -5\,x+7\,z\le 37,\ 6\,x-7\,z\le -18],\ [x,\ y,\ z]],\ [[y=27],\ [-z\le -15,\ z\le 16,\ -4\,x+3\,z\le -12,\ 6\,x-7\,z\le -18,\ 3\,x+z\le 61],\ [y,\ x,\ z]],\ [[y=33,\ z=18,\ x=18],\ [\ ],\ [x,\ y,\ z]],\ [[y=26],\ [-z\le -14,\ z\le 15,\ -4\,x+3\,z\le -11,\ 6\,x-7\,z\le -18,\ 3\,x+z\le 59],\ [y,\ x,\ z]],\ [[y=25],\ [-z\le -14,\ z\le 15,\ -4\,x+3\,z\le -10,\ 6\,x-7\,z\le -18,\ 3\,x+z\le 57],\ [y,\ x,\ z]],\ [[y=26],\ [-z\le -14,\ z\le 15,\ -4\,x+3\,z\le -11,\ 6\,x-7\,z\le -18,\ 3\,x+z\le 59],\ [y,\ x,\ z]],\ [[y=27],\ [-z\le -15,\ z\le 16,\ -4\,x+3\,z\le -12,\ 6\,x-7\,z\le -18,\ 3\,x+z\le 61],\ [y,\ x,\ z]],\ [[y=25,\ z=15,\ x=14],\ [\ ],\ [x,\ y,\ z]],\ [[y=25],\ [-z\le -13,\ z\le 15,\ -4\,x+3\,z\le -11,\ 6\,x-7\,z\le -16,\ 3\,x+z\le 57],\ [y,\ x,\ z]],\ [[y=31,\ z=17,\ x=17],\ [\ ],\ [x,\ y,\ z]],\ [[y=30,\ z=16,\ x=16],\ [\ ],\ [x,\ y,\ z]],\ [[y=25],\ [-z\le -13,\ z\le 15,\ -4\,x+3\,z\le -11,\ 6\,x-7\,z\le -16,\ 3\,x+z\le 57],\ [y,\ x,\ z]],\ [[z=-2\,x+2\,y-12],\ [-y\le -25,\ y\le 34,\ -10\,x+7\,y\le 51,\ y-x\le 15,\ x\le 19],\ [x,\ y,\ z]]]$$